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VIBRATION OF A SYSTEM OF MASSIVE STAMPS ON A LINEARLY DEFORMABLE FOUNDATION*

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A variational-difference method is applied to solve dynamic contact problems with contact domains of arbitrary planform, where the tedium in realizing this method is reduced substantially while the convergence is improved by the selection of delt-like functions of a special kind as coordinate functions. Results of numerical investigations are presented for the vibrations of a system of massive rectangular stamps on an elastic bed. The presence of resonance frequencies whose values depend on the size and mass of the stamps, and the presence of a shielding effect when a surface wave runs over the system of stamps are clarified.

The vibrations of a massive foundation of rectangular planform were investigaged earlier /1/, as were the vibrations of a system of two massive rectangular foundations /2/. However, the approaches used possess worse convergence, compared with the variational-difference method,**(**Gol'dshtein R.V., Klein I.S. and Eskin G.I., A variational-difference method of solving certain integral and integro-differential equations of three-dimensional elasticity problems. Preprint No.33, Inst. Problems of Mechanics, USSR Academy of Sciences, Moscow, 1973) and require the evaluation of double integrals of strongly oscillating functions. As noted in /3/ in particular, this disadvantage is inherent in all methods based on the partition of the contact domain into cells with a uniform stress distribution therein.

1. An elastic linearly deformable foundation is considered, the occupies the volume $-\infty \leqslant x, y \leqslant \infty$, $-\infty \leqslant -h \leqslant z < 0$, on whose surface $(z = 0)$ are N massive stamps with a flat base occupying the domain $\Omega = \Omega_1 \cup \Omega_2 \ldots \cup \Omega_N$. The boundaries of the domains Ω_k are piecewise-smooth with angular points.

Certain given harmonic loads $\mathbf{f}_k(x, y)\, e^{-i\omega t}$, $k = 1, \ldots, N$ act on the stamps, and moreover, their vibrations can be caused by waves arriving from a load $\mathbf{g}(x, y)\, e^{-i\omega t}$ applied directly to the surface of the medium, where ω is the angular frequency of the steady vibrations, and t is the time. In the general case the stamps adhere to the medium, i.e.

$$\mathbf{u}(x, y, z) = \mathbf{u}_k(x, y), \quad (x, y) \in \Omega_k, \quad z = 0 \tag{1.1}$$

($\mathbf{u}, \mathbf{u}_k$ are the complex amplitudes of the displacement of the medium and the stamps). Compliance with the energy radiation conditions assuring uniqueness of the solution of the problem is required at infinity /4/.

The surface stresses $\mathbf{q}(x, y)\, e^{-i\omega t} = \{\tau_{xz}, \tau_{yz}, \sigma_z\}$ and the Lamé coefficients λ, μ of the medium are referred to the characteristic value of the shear modulus μ_0, the density ρ to the characteristic density ρ_0, and the linear dimensions to the characteristic linear dimension l (to the thickness h of the layer in numerical examples). In this case the generalized frequency $\bar{\omega} = \omega l \sqrt{\rho_0/\mu_0}$ is taken as the frequency, the forces are given in units of $\mu_0 l^2$, and the masses in $\rho_0 l^3$. All the expressions are later presented in dimensionless form; and the bar over the $\bar{\omega}$ is omitted.

The dependence of the displacement $\mathbf{u}$ of the medium on the surface loads $\mathbf{q}$ characterizing its compliance, is derived by using the standard Fourier integral transform technique, and has the form /4/

$$\mathbf{u}(x, y, z) \iint_{\Omega} \mathbf{k}(x-\xi, y-\eta, z)\, \mathbf{q}(\xi, \eta)\, d\xi\, d\eta \tag{1.2}$$

$$\mathbf{k}(x, y, z) = \frac{1}{4\pi^2} \iint_{\Gamma_1\Gamma_2} \mathbf{K}(\alpha_1, \alpha_2, z) \exp[-i(\alpha_1 x + \alpha_2 y)]\, d\alpha_1\, d\alpha_2$$

$$\mathbf{K}(\alpha_1, \alpha_2, z) = \begin{Vmatrix} -i(\alpha_1^2 M + \alpha_2^2 N) & -i\alpha_1\alpha_2(M-N) & -i\alpha_1 P \\ -i\alpha_1\alpha_2(M-N) & -i(\alpha_1^2 N + \alpha_2^2 M) & -i\alpha_2 P \\ \alpha_1 S & \alpha_2 S & R \end{Vmatrix}$$

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Here Γ_1, Γ_2 are the contours of integration whose shape is dictated by the ultimate absorption principle /5/, M, N, P, R, S are functions of $\alpha = \sqrt{\alpha_1^2 + \alpha_2^2}$ and z, determined from certain boundary value problems for systems of differential equations. In the case of homogeneous media, these functions can be written down explicitly, while numerical methods have been developed for their construction for stratified and multilayered media.* (*See /5/ and also Glushkov E.V. and Glushkova N.V., Energy computation of elastic waves excited by surface sources in a stratified space. Rostov-on-Don, 1981. Deposited in VINITI, December 24, 1981, No.5827-81).

If the stamps are non-deformable, then the displacements $\mathbf{u}_k(x, y)$ of each of them will satisfy the equations of the motion of a rigid body

$$\begin{cases} \mathbf{u}_k = \mathbf{w}_k + \boldsymbol{\varphi}_k \times \mathbf{R}_k \\ -\omega^2 m_k \mathbf{w}_k = \mathbf{P}_k, \quad k = 1, 2, \ldots, N \\ -\omega^2 J_{k,l} \varphi_{k,l} = M_{k,l}, \quad l = 1, 2, 3 \end{cases} \tag{1.3}$$

$$\mathbf{R}_k = \uparrow \{x - x_k, y - y_k, z - z_k\}$$

$$\mathbf{P}_k = \iint_{\Omega_k} (\mathbf{f}_k - \mathbf{q})\, dx\, dy$$

$$\mathbf{M}_k = \iint_{\Omega_k} \mathbf{R}_k \times (\mathbf{f}_k - \mathbf{q})\, dx\, dy$$

Here m_k is the mass, $\mathbf{w}_k$ is the displacement of the centre of mass of the k-th stamp, (x_k, y_k, z_k) are the coordinates of its centre of mass, φ_k is the vector of the angles of stamp rotation around axes passing through its centre of mass and parallel to the Ox, Oy, Oz axes, J_k is the vector of the moments of inertia with respect to these same axes, $\mathbf{P}_k$ is the principal vector, $\mathbf{M}_k$ is the principal moment of the forces acting on the k-th stamp, $-\mathbf{q}$ is the reaction of the medium to insertion of the stamp, and $J_{k,l}, M_{k,l}, \varphi_{k,l}, w_{k,l}$ $(l = 1, 2, 3)$ are the components of the corresponding vectors.

Each vector $\mathbf{u}_k$ is expressed in terms of six unknowns $w_{k,l}, \varphi_{k,l}$ $(l = 1, 2, 3)$

$$\mathbf{u}_k = \sum_{l=1}^{3} (w_{k,l}\mathbf{e}_l + \varphi_{k,l}(\mathbf{e}_l \times \mathbf{R}_k))$$

$$\mathbf{e}_1 = \begin{Vmatrix} 1 \\ 0 \\ 0 \end{Vmatrix}, \quad \mathbf{e}_2 = \begin{Vmatrix} 0 \\ 1 \\ 0 \end{Vmatrix}, \quad \mathbf{e}_3 = \begin{Vmatrix} 0 \\ 0 \\ 1 \end{Vmatrix}$$

The integral relation connecting the stamp displacements with the surface stresses has the following form by virtue of boundary conditions (1.1):

$$\mathbf{K}\mathbf{q} \equiv \iint_{\Omega} \mathbf{k}(x - \xi, y - \eta, 0)\, \mathbf{q}(\xi, \eta)\, d\xi\, d\eta = \sum_{k=1}^{N} \mathbf{u}_k(x, y) - \mathbf{u}_H(x, y), \quad (x, y) \in \Omega \tag{1.4}$$

($\mathbf{u}_H$ is the displacement of the surface caused by the load g). By (1.4)

$$\mathbf{q} = \sum_{k=1}^{N} \sum_{l=1}^{3} (w_{k,l}\mathbf{q}_{k,l} + \varphi_{k,l}\mathbf{q}_{k,l+3}) - \mathbf{q}_H \tag{1.5}$$

The vectors $\mathbf{q}_{k,m}$ $(m = 1, \ldots, 6)$, $\mathbf{q}_H$ satisfy the system of integral equations

$$\mathbf{K}\mathbf{q}_{k,l} = \chi_k \mathbf{e}_l, \quad \mathbf{K}\mathbf{q}_{k,l+3} = \chi_k(\mathbf{e}_l \times \mathbf{R}_k), \quad l = 1, 2, 3 \tag{1.6}$$

$$\mathbf{K}\mathbf{q}_H = \mathbf{u}, \quad k = 1, 2, \ldots, N$$

$$\chi_k(x, y) = \begin{cases} 1, & (x, y) \in \Omega_k \\ 0, & (x, y) \notin \Omega_k \end{cases}$$

The $6N$ unkowns $w_{k,l}, \varphi_{k,l}$ are determined from the system of linear algebraic equations obtained after substituting (1.5) into (1.3)

$$\mathbf{E}_k \mathbf{s}_k + \sum_{j=1}^{N} \mathbf{R}_{kj}\mathbf{s}_k = \mathbf{T}_k, \quad k = 1, 2, \ldots, N \tag{1.7}$$

$$\mathbf{s}_k = \uparrow \{w_{k,1}, w_{k,2}, w_{k,3}, \varphi_{k,1}, \varphi_{k,2}, \varphi_{k,3}\}$$

$$\mathbf{E}_k = -\omega^2 \operatorname{diag}(m_k, m_k, m_k, J_{k,1}, J_{k,2}, J_{k,3})$$

$$\mathbf{R}_{kj} = \begin{Vmatrix} \mathbf{A}_{1,kj} \cdots \mathbf{A}_{6,kj} \\ \mathbf{B}_{1,kj} \cdots \mathbf{B}_{6,kj} \end{Vmatrix}$$

is a 6 X 6 matrix, and

$$\mathbf{A}_{m,kj} = \iint \mathbf{q}_{j,m}\,dx\,dy,\quad \mathbf{B}_{m,kj} = \iint (\mathbf{R}_k \times \mathbf{q}_{j,m})\,dx\,dy,$$
$$m = 1, 2, \ldots, 6$$
$$\mathbf{T}_k = \left\| \begin{array}{l} \iint (\mathbf{f}_k + \mathbf{q}_H)\,dx\,dy \\ \iint (\mathbf{R}_k \times (\mathbf{f}_k + \mathbf{q}_H))\,dx\,dy \end{array} \right\|$$

(the integrals are taken over the domain Ω_k).

The systems (1.6) are solved by a variational-difference method /6/. The general scheme of the method is described in detail,*(*Babeshko V.A., Glushkov E.V., Glushkova N.V. and Zinchenko Zh.F., Steady vibrations of massive objects on the surface of an elastic medium. Rostov-on-Don, 1981. Deposited in VINITI, January 22, 1982, No.290-82) its convergence is proved, the form of the coordinate functions is given, and it is shown how to reduce the tedium of its realization on a computer. The analysis of the contact stresses under a stamp of rectangular planform that makes friction-free contact with the medium, performed by the fictitious absorption method,**(**Babeshko V.A. and Priakhina O.D., Method of fictitious absorption in the spatial dynamical contact problems of the theory of elasticity, Rostov-on-Don, 1981. Deposited in VINITI, April 10, 1981, No.1578-81) yielded agreement with the results obtained by the variational difference method. The singularity of the contact stresses in the neighbourhood of the angular points Ω_k is studied in /7/.

2. Numerical investigations were performed for the following models. The medium is an elastic layer of unit thickness, adhering rigidly to the non-deformable foundation, and Poisson's ratio of the layer is $\nu = 0.3$ Here, in (1.2)

$$R(\alpha, 0) = -\varkappa_2^2\gamma_1(\alpha^2 \operatorname{sh}\gamma_2 \operatorname{ch}\gamma_1 - \gamma_1\gamma_2 \operatorname{sh}\gamma_1 \operatorname{ch}\gamma_2)/\Delta$$
$$M(\alpha, 0) = -i\varkappa_2^2\gamma_2(\alpha^2 \operatorname{sh}\gamma_1 \operatorname{ch}\gamma_2 - \gamma_1\gamma_2 \operatorname{sh}\gamma_2 \operatorname{ch}\gamma_1)/\Delta$$
$$N(\alpha, 0) = i \operatorname{sh}\gamma_2/(\alpha^2\gamma_2 \operatorname{ch}\gamma_2)$$
$$\Delta(\alpha) = 4\alpha^2\gamma_1\gamma_2(\alpha^2 + \gamma_2^2) - \gamma_1\gamma_2[4\alpha^4 + (\alpha^2 + \gamma_2^2)^2]\operatorname{ch}\gamma_1 \operatorname{ch}\gamma_2 + \alpha^2[(\alpha^2 + \gamma_2^2)^2 + 4\gamma_1^2\gamma_2^2]\operatorname{sh}\gamma_1 \operatorname{sh}\gamma_2$$
$$\gamma_m = \sqrt{\alpha^2 - \varkappa_m^2},\ \operatorname{Re}\gamma_m \geqslant 0,\ \operatorname{Im}\gamma_m \leqslant 0,\ m = 1, 2$$
$$\varkappa_1^2 = \frac{\rho\omega^2h^2}{\lambda - 2\mu},\quad \varkappa_2^2 = \frac{\rho\omega^2h^2}{\mu} = \overline{\omega}^2\ (\mu_0 = \mu)$$

The stamps are considered to be planar $(z_k = 0,\ \mathbf{R}_k = \{x - x_k, y - y_k, 0\})$, of rectangular planform, with dimensions 3 X 4 with respect to h. The case of friction-free contact is considered for given vertical loads (case A), and also the case of the origination of contact stresses in only a direction parallel to the Ox axis (case B, film stamp).

In case A we seek $w_{k,3}$, $\varphi_{k,1}$, $\varphi_{k,2}$, σ_z, while $w_{k,1}$, $w_{k,2}$, $\varphi_{k,3}$, τ_{xz}, τ_{yz} are identically zero; in case B $w_{k,1}$, $\varphi_{k,3}$, τ_{xz} are not zero. The integral operator in system (1.6) here becomes one-dimensional, corresponding to one element in the matrix $\mathbf{K} - K_{33}$ in case A and K_{11} in case B. Moreover, the dimensions of system (1.7) in which rows corresponding to the tangential force compoents are neglected in case A and components not parallel to Ox in case B are reduced.

We take as incoming waves

$$u_{H,3} = c_3 \sum_{r=1}^{p} \operatorname{res} R(\alpha, 0)|_{\alpha=\zeta_r} \exp(i\zeta_r x) \qquad (A) \tag{2.1}$$
$$u_{H,1} = c_1\left(\sum_{r=1}^{p} \operatorname{res} M(\alpha, 0)|_{\alpha=\zeta_r}\left(\sqrt{\zeta_r} - \frac{i}{r_0\sqrt{\zeta_r}}\right)\exp(i\zeta_r x) - \sum_{s=1}^{q} \operatorname{res} N(\alpha, 0)|_{\alpha=\zeta_s'} \frac{1}{r_0\sqrt{\zeta_s}} \exp(i\zeta_s' x)\right) \qquad (B)$$

i.e., the asymptotic expansion of the surface waves arriving from the vibrations source at a distance of $r_0 \gg 1$. Here $\zeta_r\ (r = 1, \ldots, p)$, $\zeta_s'\ (s = 1, \ldots, q)$ are real poles of the functions R, M and N, and $c_1, c_3 = \text{const}$ (in the computations $c_1 = c_3 = 1$ and $r_0 = 40$).

Remark. Real loads referred to μh^2 are quantities of the order of $10^{-6} - 10^{-12}$, consequently, in going over to specific parameters the results presented below, which have been obtained for unit forces, will be reduced by many orders of magnitude.

In Fig.1 we show the zeros (dashed lines) and poles (solid lines) of the function R, M, N as a function of frequency. The frequencies for which the poles become non-eliminable and two-fold (the frequencies of the layer natural vibrations) are marked with asterisks here and henceforth ($\omega = 2.88$, 2.92 in case A, and $\omega = 1.57, 2.88$ in case B).

In Fig.2 we show the dependence of the amplitude of the forces P occurring under the stamp vibrating translationally with unit amplitude, on ω in case A (line *1*) and B (line *2*): the amplitudes of the forces under a stamp at rest alongside a vibrating stamp are supported by dashes. It is seen that at the natural frequencies of the layer that correspond to double poles equal to zero, the amplitude of the forces vanishes. This result agrees with that obtained earlier for a strip stamp /8/ and satisfies the demands of theory because of a well-known theorem (/4/, p.239).

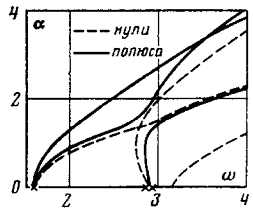

Fig.2

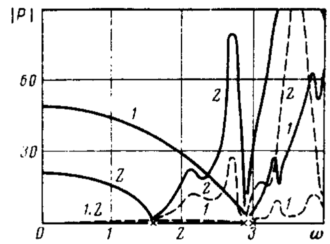

Fig.3

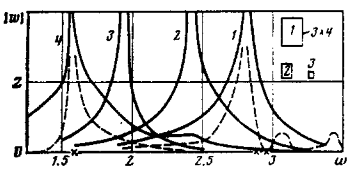

Fig.4

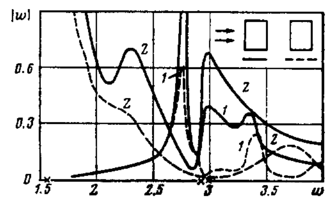

Fig.5

The solid lines *1*—*3* in Fig.3 are the amplitudes of vertical displacements of the centre of mass w_3 (case A) of a single stamp of unit mass of different sizes vibrating under the effect of unit load (the number of the line corresponds to the number of the stamp). The results of analyzing the vibrations of a system of two stamps of size 3 X 4, one of which is loaded (solid lines) while the other is load-free (dashed lines) are also shown. The displacements of the loaded stamp are practically in agreement with the displacements of a single stamp (line *1*, case A); line *4* is the result for a system of film stamps (case B).

Resonance frequencies whose location changes as the stamp size or mass changes can be seen. The existence of such discrete resonance frequencies of the stamp-layer system in the frequency range prior to the appearance of a continuous spectrum is shown in /9/. It is clear that only those frequencies at which the determinant of the system (1.7) vanishes can be resonant.

The displacements of two unloaded stamps subjected to incoming surface waves are shown in Fig.4; the solid lines in cases A (line *1*) and B (line *2*) are for the left stamp, the dashes for the right stamp. It can be seen that the screening effect, the decrease in vibration amplitude of the right stamp as compared with the vibrations of the left, is stronger at high frequencies, starting especially with the first natural frequency. For it to appear it is obviously necessary that the sizes of the overlapping domains be sufficiently large compared witb the wavelength. Thus, for stamps of one-fourth the size (0.75 X 1), screening at these frequencies is practically unnoticeable.

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STABILITY OF ANNULAR PLATES OF INHOMOGENEOUSLY AGEING VISCOELASTIC MATERIAL*

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A thin-plate bending equation in a polar coordinate system is derived for an inhomogeneously ageing material using creep theory. This equation is used to prove the sufficient condition forthe stability of annular plates by an energy method. The case of rigid clamping of both plate edges and compressive forces of dissimilar intensities along these edges is examined. Stresses in the plane of the plate are estimated, whereupon a bound is obtained on the compressible force in explicite form. An extension is made to other kinds of plate support.

Equations for the deflection and sufficient conditions for the stability of inhomogeneously ageing viscoelastic rods were obtained earlier in the one-dimensional case /1/.

1. Formulation of the problem. The strain of an annular plate of constant thickness h and radii R_0 and R $(R_0 < R)$ fabricated from an inhomogeneously ageing viscoelastic material is considered. We introduce a cylindrical system of coordinates $Or\varphi z$ with origin at the centre of the middle plane of the plate in the undeformed state and the Oz axis perpendicular to this plane.

We assume the modulus of instantaneous elastic strain E and Poisson's ratio ν of the plate material to be constant and a load consisting of a transverse distributed load of intensity $q(r,\varphi)$ and compressive forces of intensity p_0 and p on the inner and outer edges of the plate, respectively, to be applied to the plate at the time $t=0$. We let $\rho(r,\varphi)$ denote the growth of an element of viscoelastic plate material in the neighbourhood of a point with the coordinates r,φ at the time of application of an external load, and L is an operator governing the ageing properties of the material, i.e., /1/

$$Lw(t,r,\varphi)=\int_0^t L(t+\rho(r,\varphi),\tau+\rho(r,\varphi))w(\tau,r,\varphi)\,d\tau$$

where $L(t,\tau)$ is the creep kernel. The inverse operator to $I+L$ is denoted by $I-N$: $I-N=(I+L)^{-1}$, where the operator N has the same form as the operator L and governs the relaxation property of the material; the integrand $N(t,\tau)$ is called the relaxation kernel.

Let the following properties of the creep and relaxation kernels be satisfied.

1°. Functions $L_1(t,\tau)$, $N_1(t,\tau)$ exist such that for any $(r,\varphi)\in[R_0,R]\times[0,2\pi]$, $\tau\in[0,t]$ the inequalities

$$0\leqslant L(t+\rho(r,\varphi),\tau+\rho(r,\varphi))\leqslant L_1(t,\tau),\ 0\leqslant N(t+\rho(r,\varphi),\tau+\rho(r,\varphi))\leqslant N_1(t,\tau)$$

are satisfied.

2°.
$$|L_1|=\sup_t\int_0^t L_1(t,\tau)\,d\tau<\infty,\quad |N_1|<1$$

3°. A function $N_0(t,\tau)$ exists for all $\varepsilon>0$ such that starting at a certain time $t_0=t_0(\varepsilon)>0$ for all $t\geqslant\tau\geqslant t_0$

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